Assignment 4 for MATH4220

March 3,2016

You don't need to turn in the following problems:
Exercise 4.1: 2,4,5
Exercise 4.2: 1,2
Exercise 4.3: 1,3,5,6,11,16,17

Please turrn in the following 10 problems:(total points:20, due: March 24)
Exercise 4.1: 6
Exercise 4.2: 3
Exercise 4.3: 2, 4, 7, 9, 10, 12, 15
Problem 10. Solve the following problem by using separation of variables

$$\begin{cases} u_{tt} + a^2 u_{xxxx} = 0, 0 < x < l, t > 0 \\ u(x,0) = \phi(x), u_t(x,0) = \psi(x) \\ u(0,t) = u_{xx}(0,t) = u(l,t) = u_{xx}(l,t) = 0 \end{cases}$$

Exercise 4.1

2. Consider a metal rod ($0 < x < l$), insulated along its sides but not at its ends, which is initially at temperature$= 1$. Suddenly both ends are plunged into a bath of temperature$= 0$. Write the differential equation, boundary conditions, and initial condition. Write the formula for the temperature $u(x,t)$ at later times. In this problem, *assume* the infinite series expansion

$$1 = \frac{4}{\pi}\left(\sin\frac{\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \frac{1}{5}\sin\frac{5\pi x}{l} + \cdots\right).$$

4. Consider waves in a resistant medium that satisfy the problem

$$u_{tt} = c^2 u_{xx} - r u_t \qquad \text{for } 0 < x < l$$

$$u = 0 \qquad \text{at both ends}$$

$$u(x,0) = \phi(x) \qquad u_t(x,0) = \psi(x),$$

where r is a constant, $0 < r < 2\pi c/l$. Write down the series expansion of the solution.

5. Do the same for $2\pi c/l < r < 4\pi c/l$.

6. Separate the variables for the equation $tu_t = u_{xx} + 2u$ with the boundary conditions $u(0,t) = u(\pi,t) = 0$. Show that there are an infinite number of solutions that satisfy the initial condition $u(x,0) = 0$. So uniqueness is false for this equation!

Exercise 4.2

1. Solve the diffusion problem $u_t = ku_{xx}$ in $0 < x < l$, with the mixed boundary conditions $u(0,t) = u_x(l,t) = 0$.

2. Consider the equation $u_{tt} = c^2 u_{xx}$ for $0 < x < l$, with the boundary conditions $u_x(0,t) = 0$, $u(l,t) = 0$ (Neumann at the left, Dirichlet at the right).

   (a) Show that the eigenfuntions are $\cos[(n+\frac{1}{2})\pi x/l]$.

   (b) Write the series expansion for a solution $u(x,t)$.

3. Consider diffusion inside an enclosed circular tube. Let its length(circumference) be $2l$. Let x denote the arc length parameter where $-l \leq x \leq l$. The the concentration of the diffusing substance satisfies

$$u_t = k u_{xx} \qquad \text{for} \qquad -l \leq x \leq l$$

$$u(-l,t) = u(l,t) \qquad \text{and} \qquad u_x(-l,t) = u_x(l,t).$$

These are called *periodic boundary conditions.*

(a) Show that the eigenvalues are $\lambda = (n\pi/l)^2$ for $n = 0, 1, 2, 3, \ldots$.

(b) Show that the conccentration is

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} \left(A_n \cos\frac{n\pi x}{l} + B_n \sin\frac{n\pi x}{l} \right) e^{-n^2\pi^2 kt/l^2}.$$

Exercise 4.3

1. Find the eigenvalues graphically for the boundary conditions

$$X(0) = 0, \quad X'(l) + aX(l) = 0.$$

Assume that $a \neq 0$.

2. Consider the eigenvalue problem with Robin BCs at both ends:

$$-X'' = \lambda X$$

$$X'(0) - a_0 X(0) = 0, \quad X'(l) + a_l X(l) = 0.$$

(a) Show that $\lambda = 0$ is an eigenvalue if and only if $a_0 + a_l = -a_0 a_l l$.

(b) Find the eigenfunctions corresponding to the zero eigenvalue.(*Hint:*First solve the ODE for $X(x)$. The soluions are not sines or cosines.)

3. Derive the eigenvalue equation (16) for the negative eigenvalues $\lambda = -\gamma^2$ and the formula (17) for the eigenfunctions.

4. Consider the Robin eigenvalue problem. If

$$a_0 < 0, \quad a_l < 0 \quad \text{and} \quad -a_0 - a_l < a_0 a_l l,$$

show that there are *two* negative eigenvalues. This case may be called "substantial absorption at both ends."(*Hint:* Show that the rational curve $y = -(a_0 + a_l)\gamma/(\gamma^2 + a_0 a_l)$ has a single maximum and crosses the line $y = 1$ in two places. Deduce that it crosses the tanh curve in two places.)

5. In Exercise 4 (substantial absorption at both ends) show graphically that there are an infinite number of positive eigenvalues. Show graphically that they satisfy (11) and (12).

6. If $a_0 = a_l = a$ in the Robin problem, show that:

(a) There are *no* negative eigenvalues if $a \geq 0$, there is *one* if $-2/l < a < 0$, and there are *two* if $a < -2/l$.

(b) Zero is an eigenvale if and only if $a = 0$ or $a = -2/l$.

7. If $a_0 = a_l = a$, show that as $a \to +\infty$, the eigenvalues tend to the eigenvalues of the Dirichlet problem. That is,

$$\lim_{a \to \infty} \left\{ \beta_n(a) - \frac{(n+1)\pi}{l} \right\} = 0,$$

where $\lambda_n(a) = [\beta_n(a)]^2$ is the $(n+1)$st eigenvalue.

9. On the interval $0 \leq x \leq l$ of length one, consider the eigenvalue problem

$$-X'' = \lambda X$$

$$X'(0) + X(0) = 0 \quad \text{and} \quad X(l) = 0$$

(absorption at one end and zero at the other).

   (a) Find an eigenfunction with eigenvalue zero. Call it $X_0(x)$.
   (b) Find an equation for the positive eigenvalues $\lambda = \beta^2$.
   (c) Show graphically from part (b) that there are an infinite number of positive eigenvalues.
   (d) Is there a negative eigenvalue?

10. Solve the wave equation with Robin boundary conditions under the assumption that (18) holds.

11. (a) Prove that the (total) energy is conserved for the wave equation with Dirichlet BCs, where the energy is defined to be

$$E = \frac{1}{2}\int_0^l (c^{-2}u_t^2 + u_x^2)dx.$$

   (Compare this definition with Section 2.2.)

   (b) Do the same for the Neumann BCs.
   (c) For the Robin BCs, show that

$$E_R = \frac{1}{2}\int_0^l (c^{-2}u_t^2 + u_x^2)dx + \frac{1}{2}a_l[u(l,t)]^2 + \frac{1}{2}a_0[u(0,t)]^2$$

   is conserved. Thus, while the total energy E_R is still a constant, some of the internal energy is "lost" to the boundary if a_0 and a_l are positive and "gained" from the boundary if a_0 and a_l are negative.

12. Consider the unusual eigenvalue problem

$$-v_{xx} = \lambda v \qquad \text{for } 0 < x < l$$

$$v_x(0) = v_x(l) = \frac{v(l) - v(0)}{l}.$$

   (a) Show that $\lambda = 0$ is a double eigenvalue.
   (b) Get an equation for the positive eigenvalues $\lambda > 0$.
   (c) Letting $\gamma = \frac{1}{2}l\sqrt{\lambda}$, reduce the equation in part (b) to the equation

$$\gamma \sin\gamma \cos\gamma = \sin^2\gamma.$$

   (d) Use part (c) to find half of the eigenvalues explicitly and half of them graphically.
   (e) Assuming that all the eigenvalues are nonnegative, make a list of all the eigenfunctions.
   (f) Solve the problem $u_t = ku_{xx}$ for $0 < x < l$, with the BCs given above, and with $u(x,0) = \phi(x)$.

(g) Show that, as $t \to \infty$, $\lim u(x,t) = A + Bx$ for some constants A, B, assuming that you can take limits term by term.

15. Find the equation for the eigenvalues λ of the problem

$$(\kappa(x)X')' + \lambda\rho(x)X = 0 \qquad \text{for } 0 < x < l \text{ with } X(0) = X(l) = 0,$$

where $\kappa(x) = \kappa_1^2$ for $x < a$, $\kappa(x) = \kappa_2^2$ for $x > a$, $\rho(x) = \rho_1^2$ for $x < a$ and $\rho(x) = \rho_2^2$ for $x > a$. All these constants are positive and $0 < a < l$.

16. Find the positive eigenvalues and the corresponding eigenfunctions of the fourth-order operator $+d^4/dx^4$ with the four boundary conditions

$$X(0) = X(l) = X''(0) = X''(l) = 0.$$

17. Solve the fourth-order eigenvalue problem $X'''' = \lambda X$ in $0 < x < l$, with the four boundary conditions

$$X(0) = X'(0) = X(l) = X'(l) = 0,$$

where $\lambda > 0$.(*Hint:* First solve the fourth-order ODE.)